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INVESTIGATING INTO SET VALUED - STOCHASTIC INTEGRAL EQUATIONS

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Abstract

This paper examines key aspects of set-valued stochastic integrals, focusing on their properties, types, and examples, with a particular emphasis on Generalized Set-Valued Stochastic Integral 1 and Generalized Set-Valued Stochastic Integral 2. Two theorems were established, and their applications in relation to boundedness were analyzed. For indefinite integrals, we demonstrate that their effects can extend to vector-valued contexts. Based on a theoretical analysis of the properties, types, and examples of both definite and indefinite set-valued stochastic integrals, recommendations for further applications and study are presented.

Keywords: *Stochastic integral, set-valued, definite integral, indefinite integral*

Introduction

Recently, the theory of set-valued functions has been developed quickly due to the measurements of various uncertainties arising from not only the randomness but also from the impreciseness in some situations. For example, in a finance market, we consider some stock price at time t denoted by St which is a random variable defined on the probability space $\{\Omega, F, P\}$. Owing to the quick fluctuation of the stock price from time to time or to the existence of missing data, we may not precisely know the price ($S_t(w)$. a possible model for this situation would be to gie the upper and the lower prices (I.e. a margin for the error in the observation). Then we obtain an interval ($F_t(w) = [l_t(w), u_t(w)]$, which is a special kind of set valued random variable, contains not only randomness but also impreciseness, and we assure ($S_t(w)$ is certainly in the interval. The integral of a set-valued function is an interesting and an important topic, which requires quitea bit of attention with genral applications to the mathematical economics, the control theory, etc. since the space $K(X$, is not linear, the usual concepts of the integrals in a linear space are not appropriate for that of set-valued random variables. There are several approaches on defining the integration of a set-valued function. Aumann (1965) nemployeed the set of all selections of a set-valued function to define the integral with respect to a scalar-valued measure, which is called Aumann integral. Hukhara (1967) considered formal Riemann integration into the space of all convex and compact subsets. Debreu (1967) used an embedding method to consider Bochner integral in the embedded Banach space. Based on Aumann sense, Hiai-Umegaki (1967) studied the properties of the integrals of the set-valued functions, the conditional expectations with respect to a σ – finite measure and then martingales of multivalued functions. Kisielewicz (1997) used the selections method and to define the integrals of a set-valued process as a nonempty closed subset of $L^2\{\Box,F,P,R^n\}$ but did not consider it as a set-valued stochastic process. Based on Kisielewicz's work. Kim and Kim (1999) studied some properties of the stochastic integral (lt(F) defined by

$$Lt(F)(w) = \Gamma t(w) = \left\{ \int f(s,w)dBs(f(t))t>0 \in S2(F(.)) \right\},$$

where

$F = (F_t)_{t\geq 0}$ is a set-valued stochastic process, $B = (Bt)_{t>0}$ is the real-valued Brownian motion and $S^2(F(t))$ is the family of measurable selections of F with some conditions. But unfortunately, this kind of stochastic integral is still not a set-valued stochastic process since the set of stochastic integrals of all integrable selections may not be decomposable, which is not an analog to the single-valuedstochastic integrals. Jung and Kim (2003) modified the definition in 1-dimensional Euclidean space R^+ x □),the measurability and the decomposability also are based on product σ-algebra $B(R^+)$ □F.

Objective/Motivation

The theory of set-valued function has been developed quickly due to some uncertainties in Measurable Spaces. The integral of a set-valued function is an interesting topic which requires a bit of attention such that its application in real-life situation as a solution to Initial Value Problems in almost all facet of Mathematical, Economics, Physics, Chemistry etc. could be easily used by all students aiming toward excellent performance.

Literature Review

The first paper dealing with set-valued stochastic integrals is due to B. bocsan (1987. On Wiener stochastic integrals of multifunctions). Unfortunately, the definition and some properties of such defined integrals are not quite correct. Quite different definitions of the sat-valued stochastic integrals have been independently given by F. Hiai (Multivalued Stochastic integrals and stochastic inclusions, not published) and M. Kisielewicz (1995, set-valued stochastic integral and stochastic inclusions). The set-valued stochastic integrals have been defined there as some subsets of the space ($L^2(w,\chi)$ of all square integrable random variables with values at a Hibert space χ. such defined set-valued stochastic integrals are called functional set-valued stochastic integrals.

Unfortunately, the functional stochastic integrals are not decomposable subsets of $L^2(\Box,\chi)$. Therefore, they do not define set-valued random variables with sets of all their integrable selectors equal to the functional set-valued integrals. B.K. Kim and J.H. Kim (1997, stochastic integrals of set-valued processes and fuzzy processes) did not notice that and have defined a set-valued stochastic integral as a set-valued random variable with sets of all its integrable selectors covering with the functional integral defined in Kisielewicz (1995, set-valued stochastic integrals and stochastic inclusions). In 2003, E.J. Jung and J.H. Kim have correctly defined in their paper titled on set-valued stochastic integrals, the set-valued stochastic integral by the closed decomposable hull of the stochastic integral defined in Kisielewicz (1995). Unfortunately, some properties of such integrals presented in Jung and Kim (2003) are not true and proofs of some theorems and not correct. Hence, in 2011, Kisielewicz studied some properties of set-valued integrals. L. Li and S. Li discussed the Aumann type set-valued Lebesgue integral of a set-valued stochastic process with respect to time t in 2014. Also, in 2014, Kisielewicz begins about the idea of generalized set-valued stochastic integrals and studied their properties. Michta (2014/showed that the integrableboundedness of set-valued Ito's integral is equivalent to its single valuedness, Michta, earlier in 2011, extended the notion of set-valued and fuzzy stochastic integrals to semimartingaleintegrators Michta also discussed set-valued stochastic integrals driven by two-parameter martingales and increased processes. In a separable Banach space, Zhang, Mitoma and Okazaki studied in 2012 X̶-valued stochastic integrals with respect to the Poisson random measure. A set-valued function F :

□ →K(X) is said to be measurable or weakly measurable if for any open set 0 □ X, the inverse F-1 (0) ∈ *F* such a function F is called a set-valued random variable or random set, F: □ → K(X) is called strongly measurable if for any closed set C □ X, the inverse F-1 (*C*) ϵ F.

A set-valued random variable F is said to be L(p $\geq$ 1) – integrable is nonempty. Particularly, L^1 – integrable maybe briefly said to be integrable. F is called L^P (p $\geq$ 1) – integrably bounded if there exists $h \epsilon L^P(\square, F, P; R)$ such that for all $x \epsilon$ F(w), ǁxǁ< h(*w*) almost surely.

F is LP(p $\geq$ 1) – integrably bounded if and only if ǁFǁKϵ L^P(□, R)

Set-valued Stochastic Processes

F = {Ft : t $\geq$ 0} (or denoted by F = {F(t) : t $\geq$ 0}) is called a set-valued stochastic process if for every fiexd t > 0, Ft (.) is a set-valued random variable.

F = {Ft :t$\geq$ 0} is called L^P– integrable of every F_t is integrable.

A complete filtered probability space (□, F, (Ft)t$\geq$ 0, P) satisfies the following usual conditions, that is,

(i) completeness): F_o contains all the P-null subsets of □.

(ii) nondecreasing): $F_s F_t$, for all s $\leq$ t ϵ R^+.

(iii) (right continuity): $F_t = F_{t+} = \cap F_{t+\epsilon}$
ϵ>0

Set-valued Stochastic Integrals I

For a set-valued R-process *(f(t)t$\geq$0)*, an L^P – selection of *((t)t>0)* is a real valued process S_F *(f(t)t$\geq$0) ϵ LP(R)* satisfying for every

T $\geq$ 0, f(t, w) ϵ F(t, w) almost at w ϵ □.

We denote he set of all LP – selection of the set-valued R-process

(F(t)t>0) by S_p(F(t). Let S^P_F (t) be the set of all selections.

g ϵ L^P(□, At;R)

of a random set of any fixed t $\geq$ 0

Let M(□; 2^R) be the family of all measureable set-valued functions defined on □ with values in the family of monempty closed subsets of R. for any

$(f(t)_t \geq 0)$ ϵ$L^2(Kc(R))$ and t $\geq$ 0,

the random set $l_t(f) = \int_0^t f(s)dw_s$ is called a stochastic integral of *(f(t)t$\geq$0) ϵ $L^2(Kc(R))$* with respect to a real valued Brownian motion *(w(t))t$\geq$*.

SET VALUED STOCHASTIC INTEGRALS II

Properties

Theorem

Let B = (B_t)t> be an m-dimensional F-Brownian motion, Ø = (Ø)0$\leq$t$\leq$T and φ) 0<t<T the r x m-dimensional ltô integrable set-valued processes on *P = (□, F, F, P),* a complete filtered probability space. Then,

(i) $$\left[\int_0^T \O_t dB_t\right] = L2(\square, FT; R)$$ if and only if $$\left[\left[\int_0^T \O_t dB_t\right]\right] \neq \O$$

IntS

(ii) $\int_0^T \O_t dB_t$ is convex valued if Ø is complex valued,

(iii) *if (□, F, P)* is separable, then these exists a sequence

SET-VALUED STOCHASTIC INTEGRALS I

Types

Indefinite Set-valued Stochastic Integrals let Ø = (Øt)t≥0 be an r x m-dimensional ltôintegrable set-valued stochastic process. For a given above m-dimensional F-Brownian motion by by J_B (Ø) we shall denote the indefinite set-valued stochastic integral $\left(\int_0^T Ø_t dB_t\right)$. Immediately from the definition of set=valued stochastic integrals it follows that it is a closed valued F-adapted set-valued process. In what follows, by JB(Ø)t we shall denote for every t > 0 the functional set-valued stocahastic integral of Ø on [0 t] i.e.,

t≥0

$$J_B(Ø)_t = \left\{\int_0^T \varphi r dBr : \varphi \in S_{\mathbb{F}}(Ø)\right\}$$

Now from Micheal's continuous selection theorem and the properties of subtrajectory integral of the set-valued stochastic integral $\left(\int_0^T \varphi r dBr\right)$

We get the following continuous selection theorem:

Properties

We presents now the basic propettgies of generalized set-valued stochastic integral.

Theorem

For every nonmenpty set g □ L^2(R+ x □, ∑F,$^{Rd \times m}$) we have:

(i) $\int_0^T cl\mathbb{L}(g)dB_T = \int_0^T gdBt \text{ and}$

(ii) $\int_0^T co(g)dB_T = co \int_0^T gdBt$ a.s for every t ≥ 0.

GENERALIZED SET-VALUED STOCHASTIC INTEGRALS I

Given an m=-dimensional F-adapted stochastic process $\left(\int_0^T g^p dB_t\right)_{t\geq 0}$ is called the generalized indefinite set-valued stochastic integral of g_p. We have;

$$E \left\|\int_0^t cl\mathbb{L}(g)dB_T\right\|^{<p} . E = \int_0^t \max |g_r^i|^2 dr$$

GENERALIZED SET-VALUED STOCHASTIC INTEGRAL II

Types

Theorem

Let B = (Bt)t>0 be an m-dimenstional F-Brownian motion defined on PF. For every finite set {g^1,…., g^p} □ L2(R+ x □, ∑F, $R^{d \times m}$) a set-valued stochastic process

$$\left(\int_0^t g^P dB_\tau \right)_{t \geq 0} \quad with \quad gP = co\{g^1, ---, g^p\} \text{ is continuous.}$$

Example

Given a complete filtered probability space PF = (□, F, F, P) with a filtration F = (Fi)t≥0 satisfying the usual conditions, as m-dimentional F-Brownian motion B = (Bt)t≥0 defined on PF, a fixed 0 < s < t <□ and sets K □ L(R+ x □, *B* □ F, R^d), sets-valued stochastic integrals.

$$(A)\int_s^t Kdr, \ (A)\int_s^t Kdr$$

are called the Lesbegue and Auman set-valued stochastic integrals, respectively where B denotes the Borel σ – algebra of the real line.

The Aumann stochastic integral (A) $\int_s^t$ Kdr is a set-valued mpaiing defined by setting

$$[(A)\int_s^t Kdr](x) = \{ \int_s^t \varphi(T,w)dr : \varphi \in K\}$$

for every fixed ω ∈ □. In particular, if Ø = (Øt) t≥0 is a measurable integrable bounded set-valued process then Lesbegue and Aumann integrals (A) $\int_s^t Ø_r dr$ of Ø are defined on the interval [s, t] such as abovce for *K = S(Ø),* where *S(Ø),* denotes the set of all measurable selectors of Ø. These set-valued integrals can be also defined for F-nonanticipative set-valued processes. We now give the following approximation theorem for Aumann and Lebesgue stochastic integral.

Theorem

If conditions of lemma 1.3.1 are satisfied, then for every t > 0 and every sequence.

({0 = t^r < t^r < … <t^r = t})$_{r \geq I}$ of partitions of the interval [0, t] such that limr →□ δr = 0, one has

$Lim_r \to Eh \ (\int_s^t Ør\, dr, \sum_{i=0}^{lr-1} \Delta. co(Øt_i^r)) = 0$, where

$\delta r = max_{0 \leq i < lr - 1}(t^r_{i+1} - t^r_i)$ and $\Delta t^r_{i+1} - t^r_i$ for i = 0, 1, ---, lr – 1 and every r ≥ 1

This application shows that a set-valued stochastic integral can become sert-valued sub-Martingale in an M-type 2 Banach space.

Definition

An integrable convex set-valued Ft – adapted stochastic process {ft, f5 : 5 ≤} is called a set valued Ft – Martingale if for any 0 ≤ s ≤ t it holds that $E[F_t|F_s]$ = F_s in the sense of $S_{E[Ft|Fs]}$(Fs) = S_{Fs} *(Fs).* it is called a set-valued submartinglae (supermartignale) if for any 0 ≤ s ≤ t, $E[F_t|F_s]$ □ Fs (resp. $E[F_t|F_s]$ □ Fs) in the sense of $S^1_{E[Ft|Fs]}$ *(Fs)* □ S^1_{Fs} *(Fs) (resp.* $S^1_{E[Ft|Fs]}$ *(Fs)* □ S^1_{Fs} *(Fs)*

APPLICATION II

An X̶ - valued martingale f = {f_t, F_t, t ≥ 0}. The family of all L^P – martingale selection of the set-valued stochastic process F = {f_t, F_t, t ≥ 0} if it is an Lp – selection of F = {f_t, F_t, t ≥ 0}. The family of all LP – martingale selections of F = {f_t, F_t, t ≥ 0} is denoted by MS^P(F(.)). Briefly, write MS(F) = MS^1(F(.)).

Lemma

F ∈ M (□; Kc(R)) if any only if there exist two measurable functions f, g E M(□; Kc(R) s.t. for all ω E □, f(w) < g (w) and F(w) = [f(w),g(w)].

Theorem II

Let T ∈ R+, F = [f_t, F_t, t > [0, T]} be an adapted set-valued stochastic process, and F = F = [f_t, F_t, t > [0, T]} □ L1 (□, F, P; KC (X̶)). Then THE Following statements are equivalent:

(1). F = F = [f_t, F_t, t > [0, T]} is a set-valued martingale;

(2) for any 0 ≤ s < t ≤ T, we have

S^1_{Fs} *(Fs) = cl {E[g] : g E* S^1_{Fs} *(Ft)};*

(3) for any s E [0, T],

S^1_{Fs} *(Fs) = cl {gs : (gt)* $_{tE\ [0,T]}$ *E MS(F)}.*

Real Life Application of Set Value-Integrals

Stochastic integrals have many real-life application these real-life applications are but not limited to the following

(i) It very useful in Physics for calculating the distances and velocity of moving objects.

(ii) It is useful in determining the area (both curved surface and total surface) of regular irregular shaped objects.

(iii) It is also useful in Econometry (a special aspect of Economics) especially in finding the total accumulated quantities like income over time and determining the elastic points for goods and services over time.

Conclusions

From the foregoing, the following conclusions were drawn:

(i) In some areas considered difficult in Mathematics, the application of Set Value-Integral helps to determine and solve problems relating to the following;

- ❖ Area under a curve
- ❖ The area between two curves and
- ❖ The curve of mass of a body

(ii) It is easily used to render images, develop video games in computer graphics and animation.

(iii) In Engineering, the knowledge of integral is very useful in calculating the moment of inertia of a beam's cross section and also determine shear stress and the bending moment of a given load.

Recommendations

1. Mathematics teachers are encouraged to seek for a deeper knowledge of integral in order to apply it to classroom teaching.

2. Textbook writers should include the application of set integral as a problem solving examples in some related areas.

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